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Diameter of Polyhedra: Limits of Abstraction

We investigate the diameter of a family of connected graphs $G = (V, E)$ which contains the 1-skeletons of non-degenerate polyhedra in dimension d having n facets. The vertices V of G are subsets of $\{1,\ldots,n\}$ of cardinality d and the edges E of G are such that the following connectivity condition holds:

For each u, v in V there exists a path whose intermediate verticesa all contain $u \cap v$.

The largest diameter of such a graph is denoted by $\Delta(d, n)$. In the setting of non-degenerate polyhedra, each vertex is uniquely determined by the d facets in which it is contained and there exists a path from a vertex u to a vertex v which does not leave the facets in which both u and v are contained. Thus if $\Delta_u(d, n)$ is the maximum diameter of a non-degenerate polyhedron with n facets in dimension d, then $\Delta_u(d, n) \leq \Delta(d, n)$ holds.

This graph class contains furthermore a class introduced by Kalai, who, in addition to the condition above, explicitly defines the edges E to be the pairs uv such that $|u \cap v| = d - 1$. Kalai calls this relaxation of 1-skeletons ultraconnected set systems and remarks that the asymptotically best upper bounds which are known for polyhedra can also be proved to hold in this setting. These bounds are the linear upper bound $2^{d-1}n$ in fixed dimension d by Larman and the quasipolynomial upper bound $n^{\log d + 2}$ by Kalai and Kleitman.

We observe that similar bounds also hold for $\Delta(d, n)$. More precisely we prove $\Delta(d, n) \leq 2^{d-1}n$ and $\Delta(d, n) \leq n^{\log d + 1}$.

Our main result is a superlinear lower bound on $\Delta(d, n)$, namely $\Delta(d, n) \geq dn - 3d^2\sqrt{n}$, which, for a suitable choice of d is $\Omega(n^{3/2})$. The non-trivial construction relies on disjoint covering designs and uses Hall's theorem

which characterizes the existence of perfect matchings in a bi-partite graph.

This result shows that, while this simple abstraction still offers enough features to allow the proofs of the best asymptotic upper bounds on the diameter of polyhedra to be adapted, a linear bound, as it is conjectured in the famous Hirsch conjecture, would require additional features stemming from geometry.